Constructing Brownian Motion (in other words, the duck is the animal whose shadows look like 2-dimensional ducks)

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1 Introduction

The existence of Brownian motion is not immediately obvious. Here are two ways of showing it: the Kolmogorov construction via the Wiener measure and the Lévy construction with convergence.

2 Kolmogorov Construction

The Kolmogorov-Centsov approach constructs the **Wiener measure** and proves its existence, which we will show then implies the existence of Brownian motion.

Definition 1 (Law)**.** *The **law** of a stochastic process $X : \Omega \to S^T$ is the the pushforward measure $\mu = P \circ X^{-1}$, where P is a probability measure and X^{-1} is the preimage of the S^T valued random variable X.*

Definition 2 (Wiener Measure)**.** *The **Wiener measure** is a Borel product measure μ on $C([0, \infty))$ such that $\forall n \geq 1$ and $t_1 < \cdots < t_n$, the measure*

$$\mu \circ (\pi_{t_1}, \cdots, \pi_{t_n})^{-1}$$

is a multivariate Gaussian on $(\mathbb{R}^n, B(\mathbb{R}^n))$ with mean zero and covariance determined by $t_i \wedge t_j$.

Here, π_t is the coordinate projection map for time t, i.e., $\pi_t : C([0, \infty)) \mapsto \mathbb{R}$, so $\pi_t(w) = w(t)$ where w is a continuous path. So $(\pi_{t_1}, \cdots, \pi_{t_n})$ applied to w maps:

$$w \mapsto (w(t_1), \cdots, w(t_n)).$$

Therefore,

$$(\mu \circ \pi_t^{-1})([a, b]) = \mu\{w : w(t) \in [a, b]\} = \mathbb{P}(W_t \in [a, b]) \sim N(0, t)$$

where W is a random path sampled according to μ, e.g., the law of W is μ. That is, the Wiener measure is a measure such that finite dimensional projections of continuous paths $C([0, \infty))$ are specified MVN distributions. As someone on MathOverflow writes, “This definition is analogous to describing a duck as the animal whose shadows look like 2-dimensional ducks,” but it’s valid!

First, we must prove such a measure actually exists.

2.1 Kolmogorov Strategy

We will prove the existence of the Wiener measure with the following approach:

- Let $D \subset [0,1]$ (later extend to $[0,\infty)$) be a countable dense set. Use Kolmogorov’s extension.
- Show a uniformly continuous function on D can be extended to a uniformly continuous function on $[0,1]$.
- Show $\mathbb{P}$ a.s. in w the function $t \mapsto W_t(w)$ is uniformly continuous on $t \in D$.
- Show limits of Gaussians are Gaussian.

Proof. Let $D = \mathbb{Q} \cap [0,1]$. Clearly $D \subset [0,1]$ is countable and dense in $[0,1]$. For any finite index set T of D, consider the family of probability measures defined by $\mu_{t_1,\ldots,t_n} = N(0,\Sigma)$ where $\Sigma_{ij} = t_i \wedge t_j$. Under this covariance structure, these measures are **consistent**: for any permutation σ and any measurable sets $F_i \subset \mathbb{R}$,

$$\mu_{t_{\sigma(1)},\ldots,t_{\sigma(n)}}(F_{\sigma(1)} \times \ldots \times F_{\sigma(k)}) = \mu_{t_1,\ldots,t_n}(F_1 \times .. \times F_k)$$

and for $k < n$,

$$\mu_{t_1,\ldots t_k}(F_1 \times \ldots \times F_k) = \mu_{t_1,\ldots,t_k,t_{k+1},\ldots,t_n}(F_1 \times \ldots \times F_k \times \mathbb{R}^n \times (\text{m times}) \times \mathbb{R}^n).$$

By **Kolmogorov’s extension theorem**, there exists a stochastic process $(W_t)_{t \in D}$ such that

$$\mu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = \mathbb{P}(W_{t_1} \in F_1, \ldots, W_{t_k} \in F_k)$$

i.e., μ is the fidis of W:

$$(W_{t_1}, \ldots, W_{t_n}) \sim N(0,\Sigma).$$

Let $f : D \to \mathbb{R}$ be uniformly continuous. Then, by uniform continuity in $\mathbb{R}$, the sequence $(f(t_n))$ is Cauchy for any Cauchy $(t_n) \subset D$. Furthermore, there exists a well-defined limit $\tilde{f}(t) = \lim_{n\to\infty} f(t_n), t_n \to t$, independent of choice of t_n sequence. Therefore, any such f can be extended to a unique and uniformly continuous $\tilde{f} : [0,1] \to \mathbb{R}$.

We now wish to show that, almost surely in w, $t \mapsto W_t(w)$ is uniformly continuous on $t \in D$. For $s,t \in D$,

$$W_t - W_s \sim N(0, |t-s|).$$

Then for $p \geq 2$,

$$\mathbb{E}|W_t - W_s|^p = C_p|t-s|^{p/2}.$$

By **Kolmogorov-Centsov**, since we can write

$$\mathbb{E}|W_t - W_s|^p = C_p|t-s|^{1+\epsilon},$$

there exists a modification $\tilde{W}$ of W such that its sample paths are Hölder continuous on D almost surely. This implies the desired uniform continuity.

Fix $t \in [0,1]$ and choose sequence (t_n) for $t_n \in D$ and $t_n \to t$. Define

$$\tilde{W}_t(w) = \lim_{n\to\infty} W_{t_n}(w).$$

Each W_{t_n} is Gaussian and (W_{t_n}) is Cauchy in $\mathcal{L}^2$:

$$\mathbb{E}|W_{t_n} - W_{t_m}|^2 = |t_n - t_m| \to 0.$$

The $\mathcal{L}^2$ limit of Gaussian r.v.s is Gaussian, so $\tilde{W}_t$ must be Gaussian and

$$\mathbb{E}\tilde{W}_t = 0, \mathrm{Cov}(\tilde{W}_s, \tilde{W}_t) = s \wedge t.$$

Therefore, we have shown the existence of a process $(\tilde{W}_t)_{t\in[0,1]}$ with:

- almost surely continuous paths, &
- mean 0, covariance $s \wedge t$ Gaussian fidis.

Defining $\mu = \mathbb{P} \circ \tilde{W}^{-1}$ on $C([0,1])$, we have constructed μ as the Wiener measure. □

2.2 Wiener Measure Implies Brownian Motion

Above shows existence of Wiener measure/Wiener process, and in particular, $\{W_t\}_{t\in[0,1]}$ is Brownian motion.

Extend Brownian motion from $[0,1]$ to $[0,\infty)$:

Let $W^{(k)} = (W^{(k)}(t))_{t\in[0,1]}$ for $k = 1,2,3,....$ Define

$$W(t) = \sum_{k=1}^{m-1} W^{(k)}(1) + W^{(m)}(t-m)$$

if $m \leq t < m+1$.

This is just like "gluing" together a bunch of BMs on $[0,1]$. One can easily verify that $\{W(t)\}_{t\in[0,\infty)}$ is BM.

3 Lévy Construction

The key idea of the Lévy construction is to produce a sequence of continuous processes that converges to Brownian Motion.

Two necessary technical facts from real analysis:

Theorem 1 (Uniform Convergence of Continuous Functions Implies Continuous Limit)**.** *Let* $\{f_n(t)\}_{n=1,2,...}$ *be a sequence of continuous functions on* $t \in [0,1]$ *that converge uniformly to some function* $f(t)$*, i.e.,*

$$\sup_{t\in[0,1]} |f_n(t) - f(t)| \to 0 \text{ as } n \to \infty.$$

Then $f(t)$ *must be continuous.*

Theorem 2. *Let* $\{f_n(t)\}$ *be a sequence of continuous functions on* $[0,1]$ *such that*

$$\sum_n \sup_{t\in[0,1]} |f_{n+1}(t) - f_n(t)| < \infty.$$

Then f_n *converges uniformly to some continuous function* f*.*

With these results, we will show the following steps in the Lévy strategy:

- Define a sequence $\{W_t^n\}_{t\in[0,1]}$ of continuous processes with the property that $\sum_n \sup_{t\in[0,1]} |W_t^{n+1} - W_t^n| < \infty$ a.s.
- This implies $\{W_t^n\} \to W_t$ a.s. with W_t having continuous paths.
- Check $\{W_t\}$ has the correct Gaussian finite dimensional distributions.

Proof. Let $\{Z_{k,m}\}$ be $\sim N(0,1)$. Define the functions h on $[0,1]$ such that $h_0(t) := 1$, and for $k \geq 0, m = 0, ..., 2^k - 1$,

$$h_{k,m}(t) = \begin{cases} 2^{k/2} & t \in [m/2^k, (m+1/2)/2^k] \\ -2^{k/2} & t \in [(m+1/2)/2^k, (m+1)/2^k] \\ 0 & \text{else.} \end{cases}$$

Let $\psi_{k,m}(t) = \int_0^t h_{k,m}(s)ds$. Each $\psi_{k,m}$ is continuous, piecewise linear, and

$$||\psi_{k,m}||_\infty = 2^{-k/2-1}.$$

Define

$$W_t^n = Z_0 t + \sum_{k=0}^{n} \sum_{m=0}^{2^k-1} Z_{k,m}\psi_{k,m}(t).$$

Intuitively, this is like building a bunch of Haar wavelets $h_{k,m}$ and summing integrals over them with random weights $\sim N(0,1)$. Small k corresponds to coarse randomness (fewer partitions), and large k creates finer "noise," ultimately bounded by n.

Each $(W_t^n)_{t\in[0,1]}$ is continuous. Furthermore,

$$\sup_{t\in[0,1]} |W_t^{n+1} - W_t^n| = \max_{0\leq m<2^{n+1}} |Z_{n+1,m}| ||\psi_{n+1,m}||_\infty = 2^{-n/2-3/2} \max_{m<2^{n+1}} |Z_{n+1,m}|.$$

Since $\mathbb{P}(|Z| > x) \leq e^{-x^2/2}$,

$$\mathbb{P}(\max_{m<2^{n+1}} |Z_{n+1,m}| > a) \leq 2^{n+1} e^{-a^2/2}.$$

Defining $a_n = \sqrt{4(n+1)\log 2}$ for convenience,

$$\mathbb{P}(\max_{m<2^{n+1}} |Z_{n+1,m}| > a_n) \leq 2^{-(n+1)}.$$

By Borel-Cantelli, $\sum_{n=0}^{\infty} \mathbb{P}(\max_{m<2^{n+1}} |Z_{n+1,m}| > a_n) < \infty$ implies $\max_{m<2^{n+1}} |Z_{n+1,m}| \leq a_n$ for all large n almost surely. Then we can write

$$\max_{m<2^{n+1}} |Z_{n+1,m}| \leq_{a.s.} C\sqrt{n} \text{ for large n.}$$

Then, again for large n, $\sup_{t\in[0,1]} |W_t^{n+1} - W_t^n| \leq C'\sqrt{n}2^{-n/2}$ and since

$$\sum_{n=0}^{\infty} 2^{-n/2}\sqrt{n} < \infty,$$

we have that

$$\sum_n \sup_{t\in[0,1]} |W_t^{n+1} - W_t^n| < \infty \text{ a.s.}$$

Then $\{W_t^n\} \to \{W_t\}$ uniformly almost surely, and since each W_t^n is continuous, W_t has continuous paths almost surely.

For $0 \leq t_1 < ... < t_r \leq 1$, elements of the vector $(W_{t_1}^n, ..., W_{t_r}^n)$ are built from finite linear combinations of independent Gaussians. Therefore, taking linear combinations of these elements will also be Gaussian, so the entire vector is joint Gaussian. Furthermore, since

$$(W_{t_1}^n, ..., W_{t_r}^n) \to^{\mathcal{L}^2} (W_{t_1}, ..., W_{t_r}),$$

and limits of Gaussians are Gaussian, we have that $(W_{t_1}, ..., W_{t_r})$ is Gaussian.

$$\mathbb{E}(W_t W_s) = \int_0^{t\wedge s} 1du = t \wedge s$$

by orthonormality of the Haar basis. Therefore,

$$(W_{t_1}, ..., W_{t_r}) \sim N(0, (t_i \wedge t_j)_{i,j})$$

has the proper covariance structure/fidis, and so W is Brownian motion.

$\square$